TEST 1

Davis
M212

Name:
Pledge:

Show all work; unjustified answers may receive less than full credit.

You may need the following formulas: $\cos^2(x) = \frac{1}{2} + \frac{\cos(2x)}{2}$; $\sin^2(x) = \frac{1}{2} - \frac{\cos(2x)}{2}$; $\tan(x) = \frac{\sin(x)}{\cos(x)}$; $\int \sin^n(x)dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx$.

(25pts.) **1.** Without your calculators, compute the following:

a. $\int_1^2 xe^{x^2}dx$

u-substitution with $u = x^2; du = 2xdx; \frac{1}{2}\int_{u=1}^{u=4} e^u du = \frac{1}{2}(e^4 - e^1)$.

b. $\int \arcsin(5x)dx$

Integration by parts with $u = \arcsin(5x); dv = dx; du = \frac{5}{\sqrt{1-25x^2}}; v = x; \int \arcsin(5x)dx = x\arcsin(5x) - \int \frac{5x}{\sqrt{1-25x^2}}dx$; a u-substitution on this last part yields the answer of $x\arcsin(5x) - \frac{1}{5}\sqrt{1-25x^2} + C$.

c. $\int_0^\pi e^{2t}\cos(3t)dt$

Double integration by parts. First, $u = \cos(3t); dv = e^{2t}dt; du = -3\sin(3t)dt; v = \frac{1}{2}e^{2t}$; then the second time you will need to integrate $u = \frac{3}{2}\sin(3t); dv = e^{2t}dt; du = \frac{9}{2}\cos(3t)dt; v = \frac{1}{2}e^{2t}$. Adding the $\frac{9}{4}\int e^{2t}\cos(3t)dt$ to the other side, we get $\int_0^\pi e^{2t}\cos(3t)dt = \frac{4}{13}(\cos(3t)\frac{1}{2}e^{2t} + \frac{3}{2}\sin(3t)\frac{1}{2}e^{2t})|_0^\pi = \frac{4}{13}(-\frac{1}{2}e^{2\pi} - \frac{1}{2})$.

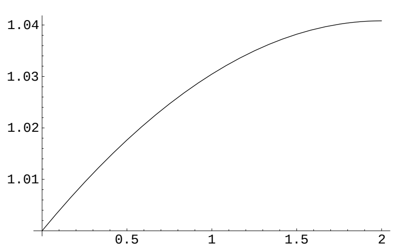
d. $\int \frac{x+3}{x^2+6x+5}dx$

Partial fractions. The denominator factors as $(x+5)(x+1)$, so we get two equations $A + B = 1; A + 5B = 3; B = \frac{1}{2}; A = \frac{1}{2}$, so $\int \frac{x+3}{x^2+6x+5}dx = \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x+1) + C$.

e. $\int \frac{x^2}{\sqrt{1-x^2}}dx$

Trig substitution. $x = \sin(t); dx = \cos(t)dt; \int \frac{x^2}{\sqrt{1-x^2}}dx = \int \frac{\sin^2(t)\cos(t)}{\cos(t)}dt = -\frac{1}{2}\sin(t)\cos(t) + \frac{1}{2}t + C = -\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin(x) + C$.

(15pts.) **2.** For the function pictured below, suppose we are trying to estimate $\int_0^2 f(x)dx$. Determine whether the following statements are true or false and give a one sentence justification:



a. $R_n \le T_n$

R_n is an overestimate since the function is increasing and T_n is an underestimate since the function is concave down, so this is false.

b. $T_n \le 1.8$

T_n will be at least as big as $L_1 = 2$, so this is false.

c. $M_n \le L_n$

M_n is an overestimate since the function is concave down and L_n is an underestimate since the function is decreasing, so this is false.

d. $\int_0^2 f(x)dx \le L_n$

L_n is an underestimate of the true answer since the function is increasing, so this is false.

e. $R_{1000} \le R_{2000}$

R_n is an overestimate, and the bigger n gets, the more accurate the estimation, so R_{2000} is smaller than R_{1000}, so this is false.

(20pts.) **3.** Estimate $\int_0^3 e^{-x^2/2}dx$ using M_4 and T_4 (show how you got your answer), and estimate the error in both cases. Is either of these an underestimate? (Explain your answer!)

Mathematica yields $M_4 = 1.25064$ and $T_4 = 1.24845$. Both errors require an estimation of the maximum value of the second derivative of the function. The second derivative is $f''(x) = (x^2-1)e^{-x^2/2}$. This has maximum value at $x = 0$ of $K_2 = 1$. Thus, the error of M_4 is no worse than $\frac{1(3)^3}{24(4)^2} = .07$ and the error of T_4 is no worse than $\frac{1(3)^3}{24(4)^2} = .14$. We don't know if either of these is an underestimate since the concavity changes over the interval (the function is concave down on $[0,1]$ and concave up on $[1,3]$).

(20pts.) **4.** Use integration by parts to justify the following formula: $\int \cos^n(x)dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx$ for n positive. Use the formula to calculate $\int \cos^6(x)dx$.

See the answers to quiz 1 for the model of how to do this (interchange sines and cosines throughout that argument and be careful of minus signs).

(20pts.) **5.** Explain the error formula for the midpoint rule as follows:

a. Compute the exact area under the curve $y = \frac{K_2x^2}{2}$ from 0 to $h = \frac{b-a}{n}$.

$\frac{K_2h^3}{6}$ by the fundamental theorem of calculus.

b. Estimate the same area as part a. using the midpoint rule with one rectangle (M_1).

The function value at the midpoint is $\frac{K_2(\frac{h}{2})^2}{2}$, and multiply this by the width h to get $M_1 = \frac{K_2h^3}{8}$.

c. Find the distance between the true area and the midpoint estimate.

$$\frac{K_2h^3}{6} - \frac{K_2h^3}{8} = \frac{K_2h^3}{24}.$$

d. Explain how the error computed in part c. is related to the error estimate over the entire interval from a to b.

The error computed in part c. is over one small interval, and the error estimate over the entire interval is n times this (there are n such intervals, all with the same error).

e. Explain in words how this procedure will work for any function $f(x)$ over any interval.

The K_2 represents the maximum value of the second derivative of $f(x)$ over the interval from a to b. The true error associated to the midpoint estimation is no worse than the error we get from assuming that the concavity is as large as it possibly can be over the entire interval.