TEST 3

Davis
M212

Name:
Pledge:

Show all work; unjustified answers may receive less than full credit.

(20pts.) **1.** Show that the sum $\Sigma_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ if $|r| < 1$. What happens if $|r| > 1$?

The partial sum for this series is $S_n = a + ar + \ldots + ar^n$; when we multiply that by r and subtract the two equations, we get $(1-r)S_n = a - ar^{n+1}$. Thus, $S_n = \frac{a(1-r^{n+1})}{1-r}$. When $n \to \infty$, the term $r^{n+1} \to 0$ as long as $|r| < 1$, which implies that $S_n \to \frac{a}{1-r}$ in that case. When $|r| > 1$, the partial sums diverge.

(20pts.) **2.** For the following series, determine whether the series converges or diverges. If the series converges, find an upper bound on its limit. Justify your answers!

a. $\Sigma_{k=2}^{\infty} \frac{k}{k^{3/2}-1}$

Compare this series to $\Sigma_{k=2}^{\infty} \frac{1}{k^{1/2}}$: this series is smaller and diverges, so the series $\Sigma_{k=2}^{\infty} \frac{k}{k^{3/2}-1}$ must also diverge.

b. $\Sigma_{k=0}^{\infty} \frac{k!}{2^k}$

The ratio test yields $\lim_{k\to\infty} \frac{\frac{(k+1)!}{2^{k+1}}}{\frac{k!}{2^k}} = \lim_{k\to\infty} \frac{k+1}{2} = \infty$. Since this is bigger than 1, the series diverges.

c. $\Sigma_{k=2}^{\infty} \frac{1}{k^4+1}$

The comparison test with $\Sigma_{k=2}^{\infty} \frac{1}{k^4}$ implies convergence. We can use $\int_1^{\infty} \frac{dx}{x^4} = \frac{1}{3}$ as an upper bound (if you do the integral from 2, you will need to use a fudge factor).

(20pts.) **3.** Find the interval of convergence (endpoint behavior too!) for:

a. $\Sigma_{n=1}^{\infty} \frac{x^n}{5^n}$

The ratio test yields $|\frac{x}{5}| < 1$, or $-5 < x < 5$. The endpoints both diverge (one is $1+1+1+\cdots$ and the other is $1-1+1-1+\cdots$).

b. $\Sigma_{n=1}^{\infty} \frac{(x+1)^n}{n}$.

The ratio test yields $\lim_{n\to\infty} |x+1| \frac{n}{n+1} = |x+1|$. To converge, we must have $|x+1| < 1$, which implies that $-1 < x+1 < 1$, or $-2 < x < 0$. The endpoint $x = -2$ converges since it is the alternating harmonic series whereas the endpoint $x = 0$ diverges since it is the harmonic series.

(20pts.) **4. a.** Use the power series for e^x to get a power series for $e^{-x^2/2}$. What is the interval of convergence for the power series for $e^{-x^2/2}$?

Since $e^x = 1 + x + x^2/2! + x^3/3! + \ldots$, we see that $e^{-x^2/2} = 1 - \frac{x^2}{2} + \frac{x^4}{4 \cdot 2!} - \frac{x^6}{8 \cdot 3!} + \cdots$. The interval of convergence is the same as the original, namely $(-\infty, \infty)$.

b. Use the answer from part a. to find a series that computes the area under the curve $e^{-x^2/2}$ on the interval [0,1].

Integrate the power series to get $x - \frac{x^3}{3 \cdot 2} + \frac{x^5}{5 \cdot 4 \cdot 2!} - \frac{x^7}{7 \cdot 8 \cdot 3!} + \cdots$ and then plug in the limits of integration to get $1 - \frac{1}{3 \cdot 2} + \frac{1}{5 \cdot 4 \cdot 2!} - \frac{1}{7 \cdot 8 \cdot 3!} + \cdots$.

c. Use the answer to part b. to find an approximation to the area under the curve $e^{-x^2/2}$ from [0,1] within .1. Justify your answer!

The answer in part b. is an alternating series, so the error in a partial sum is no worse than the next term in the series. The third term is $\frac{1}{40}$, so the estimate $1 - \frac{1}{6} = \frac{5}{6}$ is within the appropriate distance of the true answer.

(20pts.) **5.** Compute the first four nonzero terms of the MacLaurin series for the function $f(x) = (1-x)^{\frac{2}{3}}$. Use the third degree MacLaurin polynomial to approximate $f(1)$. How much error do you have in that approximation?

The first four nonzero terms are $1 - \frac{2}{3}x - \frac{2}{9\cdot 2!}x^2 - \frac{8}{27\cdot 3!}x^3$. The approximation for $f(1)$ using that polynomial is $1 - \frac{2}{3} - \frac{1}{9} - \frac{4}{81} = \frac{14}{81}$. Since the true answer is 0, we could use the $\frac{14}{81}$ as the error estimate, but you could also use $\frac{K_4 x^4}{4!}$. I was accepting lots of estimates for K_4: K_4 actually gets infinitely big when you get close to 1, so there is no actual upper bound on that value.

Have a GREAT Thanksgiving!