TEST 1

Davis
M212

Name:
Pledge:

Show all work; unjustified answers may receive less than full credit.

(18pts.) **1.** Without your calculators, compute the following:

a. $\int_0^{\sqrt{\pi/2}} x\cos(x^2)dx$

u-substitution with $u = x^2; du = 2xdx; \frac{1}{2}\int_{u=1}^{u=4}\cos(u)du = \frac{1}{2}(\sin(\pi/2) - \sin(0)) = \frac{1}{2}$.

b. $\int \ln(x+2)dx$ Start with the t-substitution $t = x+2$ to get the integral $\int \ln(t)dt$.

Now do integration by parts with $u = \ln(t)$ and $v' = 1$ to get $\int \ln(t) = t\ln(t) - \int dt = t\ln(t) - t + C$. The final answer is therefore $(x+2)\ln(x+2) - (x+2) + C$.

c. $\int \frac{x+3}{x^2+7x+6}dx$ Partial fractions. The denominator factors as $(x+6)(x+1)$, so we get

$A(x+1) + B(x+6) = x+3$. Plugging in $x = -1$ and $x = -6$ yields $A = \frac{3}{5}$ and $B = \frac{2}{5}$, so $\int \frac{x+3}{x^2+6x+5}dx = \frac{3}{5}\ln|x+6| + \frac{2}{5}\ln|x+1| + C$.

(18pts.) **2.** Verify the formula $\int x\cos(x)dx = \cos(x) + x\sin(x)$ (a) by differentiation; and (b) by using integration by parts. Use the formula to calculate $\int e^t\cos(e^t)e^t dt$. (a) $(\cos(x) + x\sin(x))' = -\sin(x) + x\cos(x) + \sin(x) = x\cos(x)$ as expected. (b) Let $u = x; v' = \cos(x)$ and the formula comes out directly. Finally, $\int e^t\cos(e^t)e^t dt = \cos(e^t) + e^t\sin(e^t) + C$ (make the u-substitution $u = e^t$).

(28pts.) **3.** For all of the following questions, we will use $f(x) = \frac{\sin^2(x)}{x^4}$.

a. Use the Trapezoid Rule with $n = 4$ to estimate $\int_1^5 f(x)dx$. $\frac{1}{2}(\frac{\sin(1)^2}{1^4} + 2\frac{\sin(2)^2}{2^4} + 2\frac{\sin(3)^2}{3^4} + 2\frac{\sin(4)^2}{4^4} + \frac{\sin(5)^2}{5^4}) = .4089$

b. Find a bound on the error you make in the estimate in part a (You may use $K_2 = 6$). The error is bounded by $\frac{K_2(b-a)^3}{12n^2} = \frac{64^3}{12(4)^2} = 2$.

c. Use the comparison test to show that $\int_1^\infty f(x)dx$ converges. Compare to $\int_1^\infty \frac{1}{x^4}dx$, which is the area under a bigger function since $\sin^2(x) \leq 1$, and the bigger function has a finite area by the p test, so the integral $\int_1^\infty f(x)dx$ also converges.

d. Explain in a few sentences how the previous computations could be used to approximate $\int_1^\infty f(x)dx$. We can use the estimate in part (a) as our estimate of the area under the curve from 1 to infinity. The error that we computed in part (b) is part of the error in using this estimate, and we could use the area under the curve from 5 to infinity of $\frac{1}{x^4}$ to bound the error of $f(x)$ over that interval. This area will be very small.

(20pts.) **4.** Determine whether the following integrals converge or diverge.

a. $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\cos(x)}dx$ $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\cos(x)}dx = \lim_{t\to\frac{\pi}{2}^-}[\int_0^t \frac{\sin(x)}{\cos(x)}dx] = \lim_{t\to\frac{\pi}{2}^-}[-\ln|\cos(x)||_0^t] = \lim_{t\to\frac{\pi}{2}^-}[-\ln|\text{co}$ $\ln|\cos(0)||]$. As t goes to $\frac{\pi}{2}$ from the left, $\cos(t)$ approaches 0 from the right, and the ln of this value goes to negative infinity, so the integral diverges.

b. $\int_1^\infty \frac{1}{x^{1/2}+e^{3x}}dx$ Since $x^{1/2} > 0$ on the interval in the integral, we have that $x^{1/2}+e^{3x} > e^{3x}$, or $\frac{1}{e^{3x}} > \frac{1}{x^{1/2}+e^{3x}}$. Since $\int_1^\infty \frac{1}{e^{3x}}dx$ converges (this needs to be shown), the comparison test tells us the the integral $\int_1^\infty \frac{1}{x^{1/2}+e^{3x}}dx$ also converges. (NOTE: you need to justify the correct inequality!)

(16pts.) **5.** You are sitting still at a stop light, and the instant the light turns green you hit the accelerator. Your velocity at time t satisfies $v = t^2$, where v is measured in meters per second and t is measured in seconds. Your friend is in the car directly next to you, and her velocity at time t satisfies $v = 2t$. Who is in the lead after 3 seconds? Make sure you use integration to justify your answer!

Your friend is going faster for the first two seconds and so will be in the lead. She is $\int_0^2(2t - t^2)dt = t^2 - t^3/3|_0^2 = 4/3$ meters ahead after 2 seconds. From 2 seconds to 3 seconds you are going faster, and you catch up by $\int_2^3(t^2 - 2t)dt = t^3/3 - t^2|_2^3 = 0 - (-4/3)$. Thus, you are tied after 3 seconds.