TEST 2

Davis
M212

Name:
Pledge:

Show all work; unjustified answers may receive less than full credit.

(25pts.) **1.** Find the volume of the solid obtained by rotating the region bounded by $x = 0, y = 0$, and $y = -\frac{h}{r}x + h$ about the y-axis using (a) the disk method and (b) the shell method.

Both of these answers are the volume of a cone, which we know is $\frac{1}{3}\pi r^2 h$. For the disk method, the general form is $V_{slice} = \pi R^2 \Delta y$, where $R = x = \frac{r}{h}(h - y)$. We integrate from $y = 0$ to $y = h$, giving the result. For the shell method, the general form is $V_{shell} = 2\pi RH\Delta x$, where $R = y = -\frac{h}{r}x + h$ and $H = x$. We integrate from $x = 0$ to $x = r$, giving the result. A good picture always helps, and in this case the picture is a line in the first quadrant going through the points $(0, h)$ and $(r, 0)$. The area in the triangle including those points and the origin is rotated about the y-axis, and that is the area we can see is a cone.

(25pts.) **2.** A bowl in the shape of a hemisphere with radius 2 feet is filled with water. (a) Given that water weighs 62.5 lb/ft^3, find the work required to pump half of the water out of the bowl. (b) What is the weight of the remaining water?

A slice of water at height y from the bottom of the bowl has volume $V_{slice} = \pi R^2 \Delta y$, where $R^2 = 2^2 - (2 - y)^2$ by the Pythagorean Theorem. This volume is in cubic feet, and we multiply that by 62.5 to get the force on the slice. To get the work on the slice, we need to multiply by the distance it is moved, which is $(2 - y)$, implying that $W_{slice} = 62.5\pi(4 - (2 - y)^2)(2 - y)\Delta y$. Turn this into an integral and integrate from $y = 1$ to $y = 2$.

$$62.5\pi \int_1^2 (4(2 - y) - (2 - y)^3)dy = 62.5\pi[-2(2 - y)^2 + \frac{1}{4}(2 - y)^4]|_1^2 = 62.5\pi\frac{7}{4}$$

To get the weight of the remaining volume, take out the factor that measures the distance travelled, namely $(2 - y)$, and integrate the rest from $y = 0$ to $y = 1$. I accepted the answer that did "half of the water that originally started in the bowl."

(25pts.) **3.** For the following series, determine whether the series converges or diverges. Justify your answers!

a. $\Sigma_{n=1}^{\infty} \frac{(-4)^n}{(11)\cdot 3^n}$

This is a geometric series with $r = \frac{-4}{3}$, which is smaller than -1 and hence the series diverges. You could also have used either the n^{th} term divergence test or the alternating series test to show divergence.

b. $\Sigma_{n=1}^{\infty} \frac{2^n n!}{(2n)!}$

Because the terms of this sum have factorials, we definitely want to use the Ratio Test.

$$L = \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}(n+1)!}{(2(n+1))!}}{\frac{2^n n!}{(2n)!}}\right|$$

The key observation is the $(2n + 2)! = (2n + 2)(2n + 1)(2n)!$, leading to the simplification:

$$= \lim_{n\to\infty} \left|\frac{2(n + 1)}{(2n + 1)(2n + 2)}\right| = 0$$

Since $L = 0$, this series converges by the Ratio Test.

c. $\Sigma_{n=6}^{\infty} \frac{1}{n \ln(n) - 5}$

This requires both the Comparison Test and the Integral Test. We first compare the series to $\Sigma_{n=6}^{\infty} \frac{1}{n \ln(n)}$ (there are plenty of other choices for what to compare with) and see that the new series is smaller than the original (this can be done by cross multiplication of the terms, which leads eventually to $-5 < 0$, which is certainly true). The new series can be shown to diverge by the Integral Test:

$$\int_6^{\infty} \frac{1}{x \ln(x)} dx = \cdots = \lim_{t \to \infty} \ln(\ln(t)) - \ln(\ln(6)) = \infty$$

Thus, the original series also diverges by the Comparison Test.

d. $\Sigma_{n=1}^{\infty} \frac{(-1)^n n}{n+3}$

(25pts.) **4.** Determine whether the following series converge absolutely, converge conditionally, or diverge. If the series converges (either absolutely or conditionally), get an estimate within .01 of the correct answer and explain in words how you know you are that close.

a. $\Sigma_{n=1}^{\infty} \frac{1}{n^7}$

This series converges absolutely by the p-test (since $7 > 1$ and there are no negative terms). In order to get an estimate within .01, we need $\int_n^{\infty} \frac{1}{x^7} dx < .01$, and this is accomplished for $n \cong 1.6$, so use $n = 2$ terms. Thus, the sum of the series is $1 + \frac{1}{128}$ within .01.

b. $\Sigma_{n=1}^{\infty} (-1)^n \frac{n!}{10^n}$

The Ratio Test shows that this series diverges since

$$L = \lim_{n \to \infty} \left| \frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}} \right| = \lim_{n \to \infty} \frac{n+1}{10} = \infty$$

c. $\Sigma_{n=1}^{\infty} (-1)^n \frac{n}{25n^2+1}$

This series converges conditionally since (i) the series without the $(-1)^n$ diverges in a comparison to a the series $\Sigma_{n=1}^{\infty} (-1)^n \frac{n}{25n^2}$, which diverges because it is harmonic; and (ii) the alternating series test shows that the actual series converges. The sum of this series will be within .01 if we add the first three terms to get $-.032$ since the fourth term is $\frac{4}{401} < .01$.