TEST 3

Davis | Name:
M212 | Pledge:

Show all work; unjustified answers may receive less than full credit.

(20pts.) **1.** Find the interval of convergence (endpoint behavior too!) for:

a. $\Sigma_{n=0}^{\infty}\frac{(x+2)^n}{n5^n}$

By the Ratio Test, you need to have $\lim_{n\to\infty}|\frac{\frac{(x+2)^{n+1}}{(n+1)5^{n+1}}}{\frac{(x+2)^n}{n5^n}}| = \lim_{n\to\infty}|\frac{(x+2)n}{(n+1)5}| = \frac{|x+2|}{5} = L < 1$, so $-5 < x+2 < 5$, or $-7 < x < 3$. When $x = -7$, the series is the alternating harmonic and hence converges. When $x = 3$ the series is the harmonic series and hence diverges. Thus, the interval of convergence is $-7 \leq x < 3$.

b. $\Sigma_{n=0}^{\infty}\frac{(x+1)^n}{n!}$

By the Ratio Test, you need to have $\lim_{n\to\infty}|\frac{\frac{(x+1)^{n+1}}{(n+1)!}}{\frac{(x+1)^n}{n!}}| = \lim_{n\to\infty}|\frac{x+1}{n+1}| = L = 0$, so this series converges for all real values of x. In fact, this is the Taylor series for e^{x+1} about $a = -1$.

c. $\Sigma_{n=0}^{\infty}\frac{(2n)!x^n}{n!}$

By the Ratio Test, you need to have $\lim_{n\to\infty}|\frac{\frac{(2(n+1))!x^{n+1}}{(n+1)!}}{\frac{(2n)!x^n}{n!}}| = \lim_{n\to\infty}|\frac{(2n+2)(2n+1)x}{n+1}| = L = \infty$ unless $x = 0$. Thus, this series converges only when $x = 0$.

(20pts.) **2. a.** Use the MacLaurin series for $\sin(x)$ to get a power series for $\sin(x^2)$. What is the interval of convergence for the power series for $\sin(x^2)$?

The MacLaurin series for $\sin(x) = \Sigma_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$, so $\sin(x^2) = \Sigma_{k=0}^{\infty}\frac{(-1)^k x^{4k+2}}{(2k+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$ The interval of convergence is all real values of x.

b. Use the answer from part a. to find a series that computes $\int_0^1 \sin(x^2)dx$.

We integrate the power series from part a and plug in 1 (plugging in 0 yields 0). This gives $\Sigma_{k=0}^{\infty}\frac{(-1)^k x^{4k+3}}{(4k+3)(2k+1)!}|_0^1 = \frac{1}{3} - \frac{1}{42} + \frac{1}{11\cdot 120} - \frac{1}{15\cdot 7!} + \cdots$

c. Use the answer to part b. to find an approximation to the area under the curve $\sin(x^2)$ from [0,1] within .001. Justify your answer!

We can use the first two terms of the series to approximate the area under the curve within .001 since the series alternates and converges and the third term is smaller than .001. Thus, the approximation is $\frac{1}{3} - \frac{1}{42} \cong .3095$.

(20pts.) **3. a.** Compute the first four nonzero terms of the MacLaurin series for the function $f(x) = (1+x)^{-\frac{1}{3}}$.

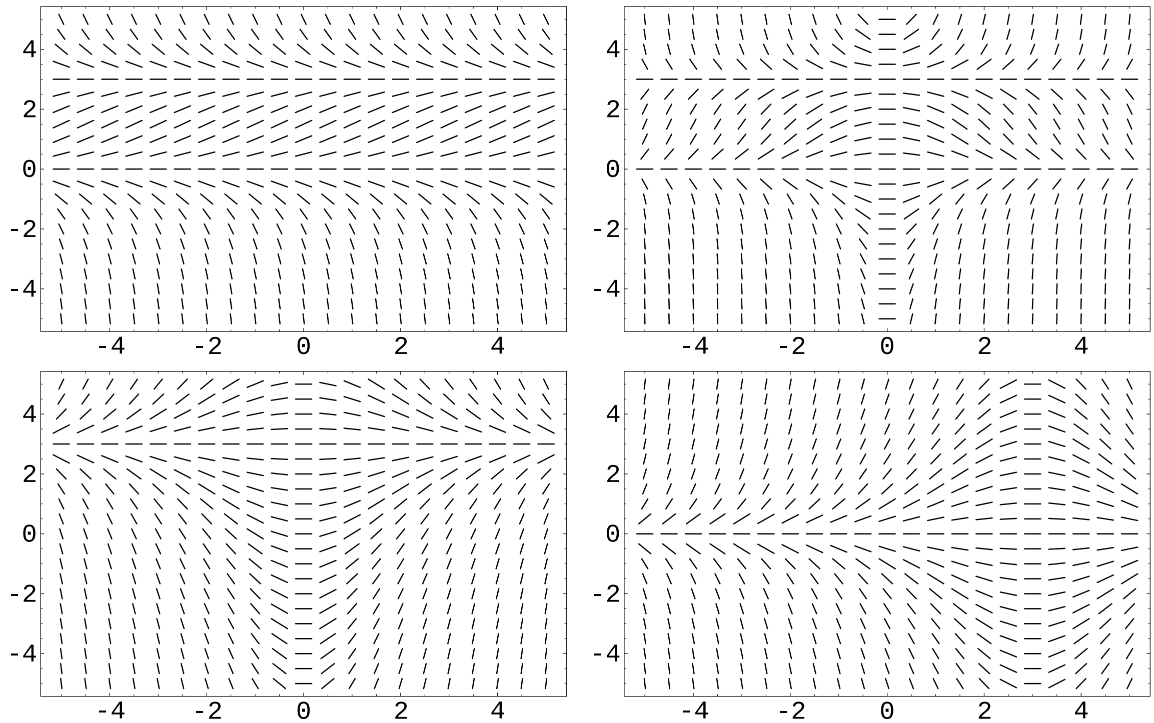
The derivatives are $f'(x) = -\frac{1}{3}(1+x)^{-\frac{4}{3}}$; $f''(x) = \frac{4}{9}(1+x)^{-\frac{7}{3}}$; $f'''(x) = -\frac{28}{27}(1+x)^{-\frac{10}{3}}$; $\ldots$; $f^{(k)}(x) = \frac{(-1)^k 1\cdot 4\cdot 7\cdots(3k-2)}{3^k}(1+x)^{-\frac{3k+1}{3}}$. Plugging in $x = 0$ and dividing by $k!$ yields coefficients $c_k = \frac{(-1)^k 1\cdot 4\cdot 7\cdots(3k-2)}{k!3^k}$. Thus, the MacLaurin series is $1 - \frac{1}{3}x + \frac{4}{18}x^2 - \frac{28}{162}x^3 + \cdots + \frac{(-1)^k 1\cdot 4\cdot 7\cdots(3k-2)}{k!3^k}x^k + \cdots$.

b. Estimate $(1.1)^{-\frac{1}{3}}$ within .1, and justify your answer.

Use $x = .1$ in the series above. You only need the first term of the series since this is alternating, so 1 is an approximation of $(1.1)^{-\frac{1}{3}}$ (the next term in the series is $-.0333$). If you used the Taylor error formula, you should use $M = 1$, and you get that you need 1 term.

(30pts.) **4.** Consider the differential equation $\frac{dy}{dx} = y(1 - x/3)$.

a. Circle the direction field associated to the differential equation. Give a reason for your answer.



The bottom right picture is the correct one. It is easiest to see by looking at where the differential equation has 0 slope, which is where $y = 0$ or $x = 3$.

b. Use Euler's method with a step size of $h = 1$ to estimate the value of $y(2)$, where $y(0) = 1$.

If we start at $y(0) = 1$, then the slope of the tangent line at that point is $\frac{dy}{dx} = 1(1-0/3) = 1$. Thus, $y_2 = 1+1(1) = 2$. We reevaluate the derivative at that point, giving $\frac{dy}{dx} = 2(1-1/3) = \frac{4}{3}$. Our estimate for $y(2)$ is $2 + 1(\frac{4}{3}) = \frac{10}{3}$.

c. Solve the differential equation and use the initial value of $y(0) = 1$ to get an exact value for $y(2)$.

The solution to the differential equation comes from separating the variables and integrating, so $\int \frac{dy}{y} = \int (1 - x/3)dx$. This implies that $\ln|y| = x - x^2/6 + C$, or $|y| = e^{x-x^2/6}e^C$. Plugging in the initial value shows that $1 = e^0 e^C$, so $e^C = 1$. Thus, our equation is $|y| = e^{x-x^2/6}$, and plugging in $x = 2$ yields $|y| = e^{2-2^2/6} = 3.7937$.

(10pts.) **5.** The half-life of cesium-137 is 30 years. Suppose we have a 100 mg sample. After how long will only 1 mg remain?

The general equation for exponential decay is $y = y_0 e^{kt}$. In this case we can use the information about half-life to figure out the value for k: $\frac{1}{2}y_0 = y_0 e^{k(30)}$, so $k = \frac{\ln(\frac{1}{2})}{30}$. If we start with 100 mg and want to know the time when 1 mg is left, we solve the equation $1 = 100e^{\frac{\ln(\frac{1}{2})}{30}t}$ for t. This gives an age just under 200 years (199.3).