TEST 2

Davis
M212

Name:
Pledge:

Show all work; unjustified answers may receive less than full credit.

(20pts.) **1.** Set up but DO NOT evaluate the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $x = y^2$ about the the line $y = 2$ using (a) the washer method and (b) the shell method. Show all your steps.

The points of intersection of these two curves are $(0, 0)$ and $(1, 1)$.

Washer method: The thickness of the washer is in the x direction, so we use Δx for the thickness. The cross-sectional area is $\pi R^2 - \pi r^2$, where R is the outer radius and r is the inner radius. We need to get both of these quantities in terms of x, and they are the distance from the axis of revolution to the functions. Thus, $R = 2 - x^2$ and $r = 2 - \sqrt{x}$. If we add up the washers, we get $\Sigma\pi((2 - x^2)^2 - (2 - \sqrt{x})^2)\Delta x$. Letting the number of washers go to ∞, we get $V = \int_0^1 \pi((2 - x^2)^2 - (2 - \sqrt{x})^2)dx$.

Shell method: The thickness of the shell is in the y direction, so we use Δy in this part for the thickness. The volume of a shell is $2\pi r h \Delta y$, where r is the radius of the shell and h is the height. The radius is the distance from the axis of revolution, so $r = 2 - y$, and the height is the distance between the two functions, so $h = \sqrt{y} - y^2$. This gives the approximation $V \cong \Sigma 2\pi(2 - y)(\sqrt{y} - y^2)\Delta y$. When we let the number of shells go to ∞, we get $V = \int_0^1 2\pi(2 - y)(\sqrt{y} - y^2)dy$.

(20pts.) **2.** An ice cream cone of radius 2 inches and height 8 inches is filled with cookie dough ice cream, which has a density of .1 pounds per square inch. Set up, but DO NOT evaluate, the integral to find the work done (in inch-pounds) to empty the cone.

Take the ice cream out a slice at a time. The volume of the slice at height y, where y is being measured from the bottom of the cone, is $\pi r^2 \Delta y$, where r is the radius of the circular cross-section. We can use similar triangles to show that $r = \frac{y}{4}$. The force of this slice is its weight, which we determine by multiplying the volume times the density, so $F_{slice} = \pi(\frac{y}{4})^2\Delta y(.1)$. Finally, the work we do to lift this slice out of the cone is the force times the distance, so we get $W_{slice} = \pi(\frac{y}{4})^2\Delta y(.1)(8 - y)$. We add the work done on all the slices up to get the approximation $W \cong \Sigma\pi(\frac{y}{4})^2(.1)(8 - y)\Delta y$. Letting the number of slices go to ∞ yields the exact answer $W = \int_0^8 \pi(\frac{y}{4})^2(.1)(8 - y)dy$.

(20pts.) **3.** For the following series, determine whether the series converges or diverges. Briefly justify your answers!

a. $\Sigma_{n=1}^{\infty} \frac{(-3)^n}{(11)\cdot\pi^n}$

This is a geometric series with $r = \frac{-3}{\pi}$: since the r value is between -1 and +1, we see that the series converges. You could also have used the alternating series test as long as you justified why the terms were going to 0.

b. $\Sigma_{n=1}^{\infty} \frac{5^n}{(2n)!}$

The ratio test is the appropriate test to use here. We get $\lim_{n\to\infty} \frac{\frac{5^{n+1}}{(2(n+1)!}}{\frac{5^n}{(2n)!}} = \lim_{n\to\infty} \frac{5}{(2n+2)(2n+1)} = 0 = L$. Since $L < 1$, the ratio test tells us that this series converges.

c. $\Sigma_{n=2}^{\infty} \frac{1}{n(\ln(n))^{\frac{3}{2}}}$

The integral test is the appropriate test here. Integrate $\int_2^{\infty} \frac{1}{x(\ln(x))^{\frac{3}{2}}}dx$ by u-substitution, where $u = \ln(x)$, yielding $\lim_{t\to\infty} \frac{-2}{\sqrt{\ln(x)}}|_2^t$. This has a finite answer, so the integral converges implying that the series also converges.

d. $\Sigma_{n=5}^{\infty} \frac{1}{\sqrt{n-2}}$

The comparison theorem is the best choice here. If we compare the terms of the series to $\frac{1}{\sqrt{n}}$, we see that $\frac{1}{\sqrt{n-2}} > \frac{1}{\sqrt{n}}$. You can either say that subtracting 2 from the denominator makes the denominator smaller and hence the fraction bigger, or you could work out the algebra. Either way, you need to justify the direction of the inequality in order to use the comparison theorem. We know that $\Sigma_{n=5}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it is a p-series with $p < 1$, and since that is the smaller series the larger series must also diverge.

(20pts.) **4.** Approximate the following within .001 of the correct answer and explain in words how you know you are that close.

a. $\Sigma_{n=1}^{\infty} \frac{1}{n^9}$

To determine how many terms of the series we need to use to get the accuracy required, we integrate $\frac{1}{x^9}$ from n to ∞ (the right hand sum picture from class helps here since the rectangles represent the remainder and they are under the curve). Doing the integration, we see that $\int_n^{\infty} \frac{1}{x^9} dx = \frac{1}{8n^8}$, and we want this to be less than .001. This will happen if $n > 1.8$, so we can use $n = 2$. The sum of the first two terms of the series is $1 + \frac{1}{2^9} = \frac{513}{512}$, and that is our approximation that we know will be within .001.

b. $\int_0^{.2} \ln(1 + x^5) dx$

We need a power series for $\ln(1 + x^5)$ in order to do this one. Taking a derivative, we get $\frac{5x^4}{1+x^5}$, and we can use our geometric series trick to get a power series for this, which is $5x^4 - 5x^9 + 5x^{14} - 5x^{19} + \cdots$. Remembering that this is the power series for the integral of the $\ln(1 + x^5)$, we integrate to get a power series for the integrand, namely $C + x^5 - \frac{x^{10}}{2} + \frac{x^{15}}{3} - \frac{x^{20}}{4} + \cdots$. We can set $x = 0$ and solve for C, yielding $C = 0$. Thus, the original problem becomes $\int_0^{.2} (x^5 - \frac{x^{10}}{2} + \frac{x^{15}}{3} - \frac{x^{20}}{4} + \cdots) dx = (\frac{x^6}{6} - \frac{x^{11}}{22} + \frac{x^{16}}{48} - \frac{x^{21}}{84} + \cdots)|_0^{.2}$. Plugging these in, we get an alternating series, and the second term is smaller than the target accuracy. Since alternating series errors are at most the next term in the series, we can use the first term in the series, namely $\frac{(.2)^6}{6}$ to estimate the sum of the series within .001.

(20pts.) **5. a.** Find a power series representation for the following series and determine the interval of convergence: $f(x) = \arctan(\frac{x}{3})$.

I would get the power series for $\arctan(x)$ by taking the derivative and finding a power series for that function. The derivative is $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$, so $\arctan(x) = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. We plug in $x = 0$ to solve for C, which is $C = 0$. This series has interval of convergence $-1 < x < 1$, and you can check that the series we get when we plug in both endpoints are alternating and hence converge by the alternating series test. Now we can substitute $\frac{x}{3}$ for x to get the power series $\frac{x}{3} - \frac{x^3}{27} + \frac{x^5}{1215} - \frac{x^7}{15309} + \cdots$ with an interval of convergence $-3 \le x \le 3$.

b. Find the interval of convergence for the following power series: $\Sigma_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

This power series represents $\sin(x)$, and we showed in class that it converges for all values of x. We did it by doing the ratio test, yielding $\lim_{\to\infty} \frac{\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{x^{2n+1}}{(2n+1)!}} = \lim_{\to\infty} \frac{x^2}{(2n+3)(2n+2)} = 0 = L$. Since $L < 1$ for all values of x, the ratio test tells us that this power series converges for all values of x.