TEST 2

Davis
M212

Name:
Pledge:

Show all work; unjustified answers may receive less than full credit.

(20pts.) **1.** Suppose a rod of length 10 with density $\delta(x) = e^{-x}$ is positioned along the positive x-axis, with its left end at the origin. Find the total mass and center of mass of the rod.

The mass is the integral of the density function, and so is $\int_0^{10} e^{-x}dx = 1 - e^{-10}$. The center of mass is the moment divided by the mass: the moment is $\int_0^{10} xe^{-x}dx$, and $\overline{x} \cong 1$.

(20pts.) **2.** A farmer has a well that is 40 feet deep. He uses a rope which weighs $2 - e^{-x}$ pounds per foot at depth x (the rope weighs more at the bottom because it gets wet). If a full bucket weighs 50 pounds, calculate the work required to pull out a full bucket.

The bucket requires 50(40) foot-pounds of work to raise to the top of the well. The rope requires $\int_0^{40} x(2 - e^{-x})dx$ of work, approximately 1599 additional foot-pounds for a total of 3599 foot-pounds of work.

(10pts.) **3.** Verify that $P = \frac{1}{1+e^{-t}}$ satisfies the logistic equation $\frac{dP}{dt} = P(1 - P)$.

$\frac{dP}{dt} = \frac{e^{-t}}{(1+e^{-t})^2}$; $P(1-P) = \frac{1}{1+e^{-t}}(1 - \frac{1}{1+e^{-t}}) = \frac{1}{1+e^{-t}}\frac{1+e^{-t}-1}{1+e^{-t}} = \frac{e^{-t}}{(1+e^{-t})^2}$

(25pts.) **4.** A six foot tall tank is completely filled with water which is leaking from a small hole in the bottom. Let $y(t)$ be the height (in feet) of the water in the tank at time t (in minutes). From a principle in engineering (known as Toricelli's law), $y(t)$ satisfies the following differential equation: $\frac{dy}{dt} = -\sqrt{y}$.

a. Find a formula for the height of the water at time t.
$\int \frac{dy}{\sqrt{y}} = \int -dt; 2\sqrt{y} = -t + C; y = 1/4(-t+C)^2; 6(4) = C^2; C = 2\sqrt{6}$.

b. How long will it take for the tank to empty completely? $0 = (2\sqrt{6} - t)^2; t = 2\sqrt{6}$

c. If the tank is initially empty and we pump water in at a constant rate of 2 feet per minute, the differential equation changes to $\frac{dy}{dt} = 2 - \sqrt{y}$. Use Euler's method with $\Delta t = .5$ to estimate the height of the water after 1 minute.
$y_1 = 0 + (2 - \sqrt{0})(.5) = 1; y_2 = 1 + (2 - \sqrt{1})(.5) = 1.5$. The height of the water will be approximately 1.5 feet after one minute.

d. Circle the slope field which corresponds to the differential equation in part c.
The upper left slope field is the correct answer.

e. Identify an equilibrium point for the differential equation in c. (what happens as $t \to \infty$)
The equilibrium point is when the derivative is 0, or $y = 4$. This is where the slope field has a horizontal tangent line. This is a stable equilibrium point.

(25pts.) **5.** The temperature loss in your house is proportional to the difference between the inside and outside temperature.

a. If T is the inside temperature at time t and the outside temperature is a constant 30 degrees F, write a differential equation which describes temperature loss.
$\frac{dT}{dt} = -k(T - 30)$.

b. Solve the differential equation in part a.
$T = 30 + Ae^{-kt}$.

c. Suppose your heating system is capable of adding 2 degrees per hour. Modify your differential equation to take this into account, and solve the new differential equation.
$\frac{dT}{dt} = 2 - k(T - 30); T = 30 + \frac{1}{k}(2 - Ae^{-kt})$.