TEST 3

Davis — Name:
M212 — Pledge:

Show all work; unjustified answers may receive less than full credit.

(20pts.) **1.** A machine in a factory produces exhaust with .01% ozone at a rate of 100 m^3/min (this means that .01% of the volume of the incoming air is ozone). The total volume of air in the factory is 25000 m^3. Assume that the ozone mixes immediately with the rest of the air, and that the mixture leaves the room at the same rate as it enters.

a. Write a differential equation for the quantity of ozone in the room at time t in minutes.

$\frac{dQ}{dt} = .01 - \frac{Q}{25000}(100) = \frac{1}{250}(2.5 - Q)$

b. Solve the differential equation assuming there was no ozone in the room initially.

$\int \frac{dQ}{2.50-Q} = \frac{1}{250}\int dt; -\ln(2.50-Q) = \frac{1}{250}t + C; 2.50 - Q = Ae^{-\frac{1}{250}t}; Q = 2.50 - Ae^{-\frac{1}{250}t}$: $A = 2.50$ by plugging $Q = 0, t = 0$ into equation.

c. People become ill when the ozone concentration reaches .005%. How long does it take for the concentration of ozone in the factory to reach this level?

$.00005(25000) = 2.50(1 - e^{-\frac{1}{250}t}) : t = 172.5$ minutes (about 3 hours).

(15pts.) **2.** A pendulum of length 50 feet makes an angle of x (radians) with the vertical. When x is small, it can be shown that, approximately, $\frac{d^2x}{dt^2} = -\frac{32}{50}x$. Solve this equation assuming that the pendulum is let go from the position where $x = x_0$. ("Let go" means that the velocity of the pendulum is initially 0. Measure t from the moment when the pendulum is released.)

$x = C_1\cos(\sqrt{\frac{32}{50}}t) + C_2\sin(\sqrt{\frac{32}{50}}t)$: plug in $t = 0$ to get $C_1 = x_0$ and $C_2 = 0$, so the final answer is $x = x_0\cos(\sqrt{\frac{32}{50}}t)$.

(20pts.) **3.** True or False (write out the word, but no reason is required).

a. If $lim_{n\to\infty}a_n = 0$, then $\sum_0^\infty a_n$ converges.

False: the harmonic series is a counterexample.

b. If $f(x) = \sum_{n=0}^\infty x^n/n^2$, then $f^{(15)}(0) = 15!/15^2$.

True: use the Taylor formula.

c. $\sum_{n=0}^\infty (-1.25)^n = \frac{1}{2.25}$.

False: the geometric sum only works for values between -1 and 1.

d. The fourth degree Taylor polynomial for $\ln(x)$ about $x = 2$ is $(x-2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} + \frac{(x-2)^4}{4}$.

False: There should be a constant term of $\ln 2$ (and other problems as well).

e. If $\sum_{n=0}^\infty C_n$ converges, then so does $\sum_{n=0}^\infty C_n(-.5)^n$.

True: If you think of the C_n as coefficients of a power series, then the radius of convergence is at least 1, so anything inside that interval will converge.

(15pts.) **4.** Compute the first 4 nonzero terms for the Taylor series of the following functions.

a. $f(t) = (1 + t^2)^{-\frac{1}{2}}$

$(1+y)^{-\frac{1}{2}} = 1 - \frac{1}{2}y + (-\frac{1}{2})(-\frac{3}{2})/2!y^2 + (-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})/3!y^3 + \cdots$: $f(t) = 1 - \frac{1}{2}t^2 + (-\frac{1}{2})(-\frac{3}{2})/2!t^4 + (-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})/3!t^6 + \cdots$

b. $g(t) = t^2e^{-t^3}$

$g(t) = t^2 - t^5 + t^8/2 - t^{11}/6 + \cdots$

(15pts.) **5.** Find the radius of convergence for the following (you don't need to check the endpoints).

a. $f(x) = \sum_{n=1}^{\infty} \frac{n^2}{2^n} x^n$.

$\lim_{n\to\infty} \frac{\frac{n^2}{2^n}}{\frac{(n+1)^2}{2^{n+1}}} = \lim_{n\to\infty} \frac{2n^2}{(n+1)^2} = 2$: the radius is $|x| < 2$.

b. $g(x) = \sum_{n=1}^{\infty} \frac{n!}{(2n)!} x^n$.

$\lim_{n\to\infty} \frac{\frac{n!}{(2n)!}}{\frac{(n+1)!}{(2(n+1))!}} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{n+1} = \infty$. This converges for all x.

(15pts.) **6.** A ball is dropped from a height of 5 feet and bounces. Each bounce is $\frac{9}{10}$ of the height of the previous bounce.

a. Find the total vertical distance that the ball has traveled when it hits the floor for the n^{th} time.

$5 + 2\sum_{i=1} n - 15(\frac{9}{10})^i = 5 + 2(5)\frac{1-\frac{9}{10}^n}{1-\frac{9}{10}}$

b. Assuming that a ball dropped from height h reaches the ground in $\frac{1}{4}\sqrt{h}$ seconds, calculate the length of time it takes for the ball to stop bouncing.

$\frac{\sqrt{5}}{4} + 2\sum_{i=1}^{\infty} \sqrt{5(\frac{9}{10})}^i = \frac{\sqrt{5}}{4} + \frac{2\sqrt{5(\frac{9}{10})}}{1-\sqrt{\frac{9}{10}}}$.