TEST 3

Davis — Name:

M212 — Pledge:

Show all work; unjustified answers may receive less than full credit.

(20pts.) **1.** Show that the sum $\Sigma_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ if $|r| < 1$. What happens if $|r| > 1$?

The partial sum S_k is $S_k = a + ar + ar^2 + \cdots + ar^k$. If we multiply this equation by r we get $rS_k = ar + ar^2 + \cdots + ar^k + ar^{k+1}$. Subtracting these equations yields $(1-r)S_k = a - ar^{k+1}$. Solving this for S_k produces $S_k = \frac{a-ar^{k+1}}{1-r}$. If $|r| < 1$, then $r^{k+1} \to 0$ when $k \to \infty$, so $lim_{k\to\infty} S_k = \frac{a}{1-r}$ in that case. If $|r| > 1$, the series will diverge since the terms don't go to 0.

(20pts.) **2.** For the following series, determine whether the series converges or diverges. Name the test that you use, and briefly show why it applies.

a. $\Sigma_{k=0}^{\infty} (-\frac{2}{3})^k$

This is a geometric series with $r = -\frac{2}{3}$, so the series converges.

b. $\Sigma_{k=2}^{\infty} \frac{k}{k^{3/2}-1}$

Compare this series to $\Sigma_{k=2}^{\infty} \frac{k}{k^{3/2}} = \Sigma_{k=2}^{\infty} \frac{1}{k^{1/2}}$ which diverges and is smaller, so the series in question diverges.

c. $\Sigma_{k=0}^{\infty} \frac{3^k}{(2k)!}$

The Ratio test is the clear choice here since there is a factorial, and the ratio test is the following: $lim_{k\to\infty} \frac{\frac{3^{k+1}}{(2k+2)!}}{\frac{3^k}{(2k)!}} = lim_{k\to\infty} \frac{3}{(2k+2)(2k+1)} = 0$. This is L, which is less than 1, so the series converges.

d. $\Sigma_{k=2}^{\infty} \frac{1}{k^4+1}$

A comparison test with $\Sigma_{k=2}^{\infty} \frac{1}{k^4}$ implies that this series converges (the comparing series is bigger and has a finite sum by the integral test).

(20pts.) **3.** Find the interval of convergence (endpoint behavior too!) for:

a. $\Sigma_{n=0}^{\infty} \frac{x^n}{n5^n}$

As with all interval of convergence questions, the Ratio test is needed. It yields $|\frac{x}{5}|$, which must be less than 1, so $-5 < x < 5$. The endpoint $x = -5$ leads to the series $\Sigma_{n=0}^{\infty} \frac{(-1)^n}{n}$, which is the alternating harmonic series and converges, and the endpoint $x = 5$ leads to the series $\Sigma_{n=0}^{\infty} \frac{1}{n}$, which is the harmonic series and diverges. Thus, the interval of convergence is $-5 \leq x < 5$.

b. $\Sigma_{n=1}^{\infty} \frac{(x+1)^n}{n^2}$.

The Ratio test leads to the inequality $|x + 1| < 1$, which is the same as $-1 < x + 1 < 1$, or $-2 < x < 0$. The endpoints -2 and 0 both lead to convergent series since $\Sigma_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent. Thus, the interval of convergence is $-2 \leq x \leq 0$.

(20pts.) **4.** **a.** Use the power series for e^x to get a power series for $e^{-x^2/2}$. What is the interval of convergence for the power series for $e^{-x^2/2}$?

The power series for e^x is $\Sigma_{k=0}^{\infty} \frac{x^k}{k!}$, so the power series for $e^{-x^2/2}$ is $\Sigma_{k=0}^{\infty} \frac{(-x^2/2)^k}{k!}$. Since the interval of convergence for the power series for e^x is ∞, that is true for the power series for $e^{-x^2/2}$ as well.

b. Use the answer from part a. to find a series that computes the area under the curve $e^{-x^2/2}$ on the interval [0,1].

If we integrate term by term we get $\Sigma_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)2^k k!}$. When we plug in the endpoints 0 and 1 we get $\frac{(-1)^k}{(2k+1)2^k k!}$.

c. Use the answer to part b. to find an approximation to the area under the curve $e^{-x^2/2}$ from [0,1] within .1. Justify your answer!

The sum in part b. is alternating, so we only need sum until the next term is smaller than .1. The third nonzero term in the sum is $\frac{1}{5(4)(2)} = .025$, so the sum of the first two terms suffices. They are 1 and $-\frac{1}{6}$ which sum to $\frac{5}{6}$, so that is our estimate.

(20pts.) **5.** Compute the first four nonzero terms of the MacLaurin series for the function $f(x) = (1-x)^{\frac{2}{3}}$. Use the third degree MacLaurin polynomial to approximate $f(\frac{1}{3})$. Estimate the error of that approximation.

In order to get the MacLaurin series, you need to compute the relevant derivatives, as seen below:

$f(x)$	$(1-x)^{\frac{2}{3}}$
$f'(x)$	$\frac{2}{3}(1-x)^{-\frac{1}{3}}(-1)$
$f''(x)$	$\frac{2}{3}(-\frac{1}{3})(1-x)^{-\frac{4}{3}}(-1)(-1)$
$f'''(x)$	$\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})(1-x)^{-\frac{7}{3}}(-1)(-1)(-1)$

We then plug 0 into each of these derivatives, yielding the numbers $1, -\frac{2}{3}, -\frac{2}{9}, -\frac{8}{27}$. Plugging these into the MacLaurin formula produces $(1-x)^{\frac{2}{3}} \cong 1 - \frac{2}{3}x - \frac{1}{9}x^2 - \frac{4}{81}x^3$. To approximate $f(\frac{1}{3})$, we simply plug $\frac{1}{3}$ into the polynomial, getting .7636. To estimate the error, we need to know the maximum value of the 4th derivative of the function. The fourth derivative is $\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})(1-x)^{-\frac{10}{3}}(-1)(-1)(-1)(-1)$, which is about 2.67. If we round the maximum value up to $M = 3$, we get a maximum error of $\frac{3}{4!}(\frac{1}{3})^4 = .0046$.