Test 1

Davis
CS222

Name:
Pledge:

(10pts.) 1. Use truth tables to determine whether the following are logically equivalent.

a. $\overline{p \vee q}; \overline{p} \wedge \overline{q}$

These are equivalent (easy truth tables)

b. $(p \to q) \to r; p \to (q \to r)$.

These are not equivalent (easy truth tables)

c. $(p \wedge q) \vee r; (r \vee q) \wedge p$

These are not equivalent (easy truth tables)

(10pts.) 2. Use the logic game to determine whether the following proposition is true or false. Explain your reasoning.

$$\forall x \forall y \exists z (x < y \to (z > x) \wedge (z < y))$$

This is a true statement. If Farley chooses $y \leq x$, then the first part of the implication is false, so the statement is true. If Farley chooses $x < y$, then you can choose $z = \frac{x+y}{2}$, and this meets the conditions.

(15pts.) 3. Define a relation on the integers by $R = \{(x, y) | 4 \text{ divides } (x - y)\}$. Prove that this relation is reflexive, symmetric, and transitive.

Reflexive: Since $x - x = 0 = 4(0)$, this implies that $(x, x) \in R$ for all values of x as required.

Symmetric: Suppose $(x, y) \in R$. This implies that $x - y = 4t$ for some t, so $y - x = 4(-t)$. Thus, $(y, x) \in R$.

Transitive: Suppose $(x, y) \in R$ and $(y, z) \in R$. This implies that $x - y = 4t$ for some t and $y - z = 4s$ for some s. Adding these equations together, we get $x - z = (x - y) + (y - z) = 4t + 4s = 4(t + s)$. Thus, $(x, z) \in R$ as required.

(15pts.) 4. Give an example of a function that is one-to-one but not onto, and justify why your function is one-to-one and why it is not onto.

Let $X = \{1, 2\}$ and $Y = \{a, b, c\}$. Define the function $f : X \to Y$ by $f(1) = a$ and $f(2) = b$. This function is 1-1 since if $f(x_1) = f(x_2)$, then $x_1 = x_2$ (you don't have two elements of X going to the same element of Y). This function is not onto since there is no element going to c.

(15pts.) 5. Prove the following: $\forall n \geq 1, \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1) \cdot (2n+1)} = \frac{n}{2n+1}$.

This is an induction proof.

Base case: $n = 1$. $\frac{1}{3} = \frac{1}{3}$

Inductive step: Suppose $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1) \cdot (2n+1)} = \frac{n}{2n+1}$.

(Show that $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1) \cdot (2n+1)} + \frac{1}{(2n+1) \cdot (2n+3)} = \frac{n+1}{2n+3}$)

$(\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1) \cdot (2n+1)}) + \frac{1}{(2n+1) \cdot (2n+3)} = (\text{inductive hypothesis}) \frac{n}{2n+1} + \frac{1}{(2n+1) \cdot (2n+3)} = \frac{n(2n+3)+1}{(2n+1) \cdot (2n+3)} = \frac{2n^2+3n+1}{(2n+1) \cdot (2n+3)} = \frac{(2n+1)(n+1)}{(2n+1) \cdot (2n+3)} = \frac{n+1}{2n+3}$

(15pts.) 6. Football scores typically increase by either 3 points (for a field goal) or 7 points (for a touchdown). Show that any score of 12 or more can be achieved by some combination of field goals and touchdowns.

This is an induction proof.

Base case(s): $12 = 4$ field goals; $13 = 1$ touchdown, 2 field goals; $14 = 2$ touchdowns.

Inductive step: Let $n \geq 15$ (we have already handled the smaller cases). Suppose that a score of k points can be achieved via touchdowns and field goals for k in the range $12 \leq k < n$. (Show that a score of n points can be achieved via touchdowns and field goals.)

Since $n \geq 15$, we have that $n - 3$ is in the range $12 \leq n - 3 < n$. By the inductive hypothesis, $n - 3$ can be achieved via touchdowns and field goals. If we attach a field goal to this, we will have a a score of n that is achieved via touchdowns and field goals as required.

(10pts.) 7.

a. Add the hexadecimal numbers $49F7 + C66$. What is this number is decimal?

Adding the one's column yields 13, which is D in hex. In the "16" column, we add F and 6 to get 21, which is 5 in hex with a carry of 1. In the "16^2" column, we add 9 and C (plus the 1 that we have carried) to get 22, which is 6 in hex with a carry of 1. The final column is 4 plus the carried 1, which is 5. Thus, the sum is $565D$. In decimal, this is $5(16^3) + 6(16^2) + 5(16) + 13 = 22109$.

b. What is the decimal number 1234 in hexadecimal?

Divide this by 16, yielding a quotient of 77 and a remainder of 2. Divide 77 by 16, yielding a quotient of 4 and a remainder of 13. Divide 4 by 16, yielding a quotient of 0 and a remainder of 4.

Thus, the number in hex is $4D2$.

(10pts.) 8. Consider the relation $R = \{(0,0), (0,1), (1,0), (1,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$ on $\{0, 1, 2, 3, 4\}$.

a. Draw the graph for R and use that to show that R is an equivalence relation.

See book for example of drawings.

b. Find the equivalence classes for each element.

$[0] = \{0, 1\} = [1]; [2] = \{2\}; [3] = \{3, 4\} = [4]$.