TEST 2

Davis — Name:

CS222 — Pledge:

Show all work; unjustified answers may receive less than full credit.

(15pts.) **1.** Write pseudo code for an algorithm that computes the greatest common divisor of two numbers a and b. Explain why your algorithm works.

See p. 135 in the book for one example.

(15pts.) **2.** Let $f_1 = 1; f_2 = 2; f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ (this is the Fibonacci sequence). Use mathematical induction to show that $\sum_{k=1}^{n} f_k^2 = f_n f_{n+1} - 1$ for $n \geq 1$.

The base case is $\sum_{k=1}^{1} f_k^2 = f_1^2 = 1^2 = 1 = f_1 f_2 - 1 = 1(2) - 1 = 1$.

Suppose that $\sum_{k=1}^{n} f_k^2 = f_n f_{n+1} - 1$. (Show that $\sum_{k=1}^{n+1} f_k^2 = f_{n+1} f_{n+2} - 1$).

Start with the left hand side of the equation: $\sum_{k=1}^{n+1} f_k^2 = \sum_{k=1}^{n} f_k^2 + f_{n+1}^2 =$ (the inductive hypothesis!) $f_n f_{n+1} - 1 + f_{n+1}^2 = f_{n+1}(f_n + f_{n+1}) - 1 = f_{n+1} f_{n+2} - 1$ (the last step uses the Fibonacci recurrence relation). This completes the induction.

(15pts.) **3.** Write pseudo code for a recursive algorithm to find the minimum of a finite sequence of numbers.

See the answer in the back of the book for number 12 of section 3.4 (the answer is on p. 546).

(15pts.) **4.** Show that $n^2 + n + 1$ is $\Theta(n^2)$ (do NOT just quote the theorem: I want the complete argument!).

In order to show this, we need to show that $n^2 + n + 1$ is $O(n^2)$ and also $\Omega(n^2)$. To show O, we need to find a constant so that $n^2 + n + 1 < C_1 n^2$. The trick here is to use the following string of inequalities: $n^2 + n + 1 < n^2 + n^2 + n^2 = 3n^2$, so $C_1 = 3$ will work. To show Ω, we need to find a constant so that $n^2 + n + 1 > C_2 n^2$. The trick for this one is the following string: $n^2 + n + 1 > n^2 + 0 + 0 = n^2$, so $C_2 = 1$ will work. Since $n^2 + n + 1$ is $O(n^2)$ and $\Omega(n^2)$, that shows that it is $\Theta(n^2)$.

(10pts.) **5.** How many eight-bit strings either begin with 00 or end with 11?

There are 64 strings that start with 00 and 64 strings that end with 11, but there are 16 strings that both start with 00 AND end with 11, so the number we are looking for is 64 + 64 - 16 = 112. (there are other ways to justify this).

(10pts.) **6.** A bridge hand consists of 13 cards chosen from an ordinary deck of cards. How many possible bridge hands are there? How many contain exactly 3 aces? How many contain exactly 7 cards in one of the four suits?

There are $C(52, 13) = \frac{52!}{39!13!} = 6.35 \times 10^{11}$ bridge hands. The number that contains exactly 3 aces is $C(4, 3)C(48, 10) = 2.62 \times 10^{10}$. The number that contain exactly 7 cards in one suit is $4C(13, 7)C(39, 6) = 2.24 \times 10^{10}$.

(10pts.) **7.** Find the coefficient of the term when the expression is expanded: $s^{14}t^{10}; (2s + 3t)^{24}$.

The coefficient will be $C(24, 14)2^{14}3^{10} = 1.9 \times 10^{15}$.

(10pts.) **8.** The 14 Computer Science courses at Podunk University are labelled with numbers between 200 and 220 (inclusive). Show that there are at least two CS courses whose numbers are exactly five apart.

This is a pigeonhole principle problem. If we put all of the course numbers as well as the course number plus five into the appropriate box, we will have 28 numbers going into 26 boxes (between 200 and 225). Thus, two must go into the same box. It can't be two from the course numbers since those are distinct, and it can't be two from the course numbers plus five since those are also distinct, so it must be one from the course numbers and the other from the course numbers plus five. This proves the claim.