TEST 2

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M235 | Pledge:

Show all work; unjustified answers may receive less than full credit.

(17pts.) **1.** Use the second order Taylor polynomial to approximate $e^{.01}\cos(-.02)$, and compare it to the actual value.

The function we will use is $f(x,y) = e^x \cos(y)$, and we will do the Taylor Polynomial about $(0,0)$. These have partial derivatives $f_x(0,0) = 1; f_y(0,0) = 0; f_{xx}(0,0) = 1; f_{xy}(0,0) = 0; f_{yy}(0,0) = -1$, so the second order Taylor polynomial is $1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2$. When we plug in $x = .01$ and $y = -.02$, we get $1 + .01 + \frac{(.01)^2}{2} - \frac{(-.02)^2}{2} = 1.00985$. The actual value is approximately 1.0098482, so we are very close.

(17pts.) **2.** Find the critical points of $f(x,y) = 2x^2 + 3xy + 4y^2 - 5x - 4y$, and classify them as local maximum, local minimum, or saddle points.

The partials are $f_x = 4x + 3y - 5$ and $f_y = 3x + 8y - 4$. We set each of these equal to 0 and solve for x and y, yielding $x = \frac{112}{92}$ and $y = \frac{1}{23}$ as the critical point. To determine whether that is a local max/min or saddle point, we compute $f_{xx} = 4; f_{yy} = 8$; and $f_{xy} = 3$. The Hessian is $4(8) - (3)^2 = 23$ and $f_{xx} > 0$, so the critical point is a local min.

(16pts.) **3.** Find the point in the plane $3x + 2y + z = 14$ closest to the origin using derivatives (HINT: maximize the SQUARE of the distance function).

The square of the distance function for a point $(x,y,z) = (x,y,14-3x-2y)$ on the plane from the origin is $f(x,y) = x^2 + y^2 + (14-3x-2y)^2$. Find critical points, so set $f_x = 2x - 6(14-3x-2y) = 0$ and $f_y = 2y - 4(14-3x-2y) = 0$. Solving this system yields $x = 3$ and $y = 2$, implying $z = 1$. This critical point is a local min (and hence an absolute min) since $f_{xx} = 2 + 18 = 20; f_{yy} = 2 + 8 = 10$; and $f_{xy} = 12$ implying the Hessian is $200 - 144 = 56 > 0$ and $f_{xx} > 0$. That minimizes the distance function, so the closest point is $(3,2,1)$.

(30pts.) **4.** Evaluate the following integrals (HINT: switching the order of integration will help in some of these cases).

a. $\int_0^1 \int_y^1 e^{x^2} dx dy$

Switching the order of integration yields $\int_0^1 \int_0^x e^{x^2} dy dx$. Doing the inner integral yields $\int_0^1 x e^{x^2}$, and using a u-substitution we get $\frac{1}{2}(e-1)$ as the answer.

b. $\int_1^2 \int_1^x e^x dy dx$

If we do this without switching the order of integration, we get $\int_1^2 (xe^x - e^x)dx$. We can do the first term by parts, giving $(x-1)e^x$, and the integral of the second term is $-e^x$. When we evaluate from 1 to 2 we get e. If we switch the order of integration, we get $\int_1^2 \int_y^2 e^x dx dy$. The inner integral is $(e^2 - e^y)$, and the outer integral is $e^2 - (e^2 - e) = e$, agreeing with the other computation.

c. $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \sqrt{z} dz dy dx$

We want to delay integrating the z as long as possible, so we switch the order of integration. If we do the x first, y second, and z last, we get

$$\int_0^1 \int_0^{1-z} \int_0^{1-y-z} \sqrt{z} dx dy dz$$

This becomes $\int_0^1 \int_0^{1-z} \sqrt{z}(1-y-z) dy dz = \frac{1}{2} \int_0^1 (\sqrt{z})(1-z)^2 dz = \frac{8}{105}$.

(20pts.) **5.** The average value of a function $f(x, y, z)$ on a region W is defined to be the triple integral $f_{ave} = \frac{\int \int \int_W f(x,y,z) dx dy dz}{\int \int \int_W dx dy dz}$. Find the average value of the function $f(x, y, z) = xyze^{(x^2+y^2)^2}$ over the region bounded by $z = 0, z = 1, x \geq 0, y \geq 0$, and $x^2 + y^2 \leq 1$. (HINT: switch to a different coordinate system and use the trig identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$)

Switch to cylindrical coordinates. In that setting, we get $\int \int \int_W f(x, y, z) dx dy dz = \int_0^1 \int_0^{\pi/2} \int_0^1 r\cos(\theta) r \sin(\theta) z e^{r^4} r dz d\theta dr = \int_0^1 \int_0^{\pi/2} \int_0^1 r^3 e^{r^4} \cos(\theta)\sin(\theta) z dz d\theta dr = \frac{1}{2} \int_0^1 \int_0^{\pi/2} r^3 e^{r^4}$ co $\frac{1}{4} \int_0^1 \int_0^{\pi/2} r^3 e^{r^4} \sin(2\theta) d\theta dr = \frac{1}{4} \int_0^1 r^3 e^{r^4} dr = \frac{1}{16}(e-1)$. We divide this by the volume of the portion of the cylinder above the region, which is $\frac{\pi}{4}$, to get the average of $\frac{e-1}{4\pi}$.