TEST 3

Davis — Name:

M235 — Pledge:

Show all work; unjustified answers may receive less than full credit.

(17pts.) **1.** Compute $\int_{\mathbf{C}}(200xz^2+36y)ds$, where $c(t)=(t^5,t^3,t)$ on the interval $0\le t\le 1$.

We need the length of $c'(t)=(5t^4,3t^2,1)$, which is $\sqrt{25t^8+9t^4+1}$. Plugging the parameterization in for x,y, and z and multiplying by this length, we get $\int_{\mathbf{C}}(200xz^2+36y)ds=\int_0^1(200t^7+36t^3)\sqrt{25t^8+9t^4+1}dt=\frac{2}{3}(25t^8+9t^4+1)^{\frac{3}{2}}|_0^1=\frac{2}{3}(34^{\frac{3}{2}}-1)$.

(17pts.) **2.** Compute $\int_{\mathbf{C}} F\cdot ds$, where $F(x,y,z)=(x+y^2,y-xy,e^z)$ and C is the curve from $(2,0,0)$ to $(0,2,0)$ going around the circle centered at the origin of radius 2 in the xy-plane.

We parameterize this curve as $c(t)=(2\cos(t),2\sin(t),0)$ on the interval $0\le t\le\frac{\pi}{2}$. Plugging this in to the line integral, we get $\int_{\mathbf{C}} F\cdot ds=\int_0^{\frac{\pi}{2}}(2\cos(t)+4\sin^2(t),2\sin(t)-4\cos(t)\sin(t),e^0)\cdot(-2\sin(t),2\cos(t),0)dt=\int_0^{\frac{\pi}{2}}(-4\cos(t)\sin(t)-8\sin^3(t)+4\cos(t)\sin(t)-8\cos^2(t)\sin(t)+0)dt=\int_0^{\frac{\pi}{2}}-8\sin(t)dt=8\cos(t)|_0^{\frac{\pi}{2}}=-8$.

(16pts.) **3.** Evaluate $\int\int_{\mathbf{S}}(x^2+y^2+z^2)dS$ across the parallelogram with vertices $(1,1,2)$, $(4,2,2)$, $(1,3,3)$, and $(4,4,3)$.

Parameterize the surface as $\Phi(u,v)=(1,1,2)+u(3,1,0)+v(0,2,1)$ for $0\le u,v\le 1$. Then $||T_u\times T_v||=||\begin{pmatrix} i & j & k\\ 3 & 1 & 0\\ 0 & 2 & 1\end{pmatrix}||=\sqrt{1^2+3^2+6^2}=\sqrt{46}$. The surface integral becomes $\int\int_{\mathbf{S}}(x^2+y^2+z^2)dS=\int_0^1\int_0^1((1+3u)^2+(1+u+2v)^2+(2+v)^2)(\sqrt{46})dudv=\sqrt{46}\int_0^1\frac{1}{9}(1+3u)^3+\frac{1}{3}(1+u+2v)^3+u(2+v)^2|_0^1dv=\sqrt{46}\int_0^1(\frac{64}{9}+\frac{1}{3}(2+2v)^3+(2+v)^2-\frac{1}{9}-\frac{1}{3}(1+2v)^3-0)dv=\sqrt{46}[7v+\frac{1}{24}(2+2v)^4+\frac{1}{3}(2+v)^3-\frac{1}{24}(1+2v)^4]|_0^1=\sqrt{46}[7+\frac{256}{24}+9-\frac{81}{24}-\frac{16}{24}-\frac{8}{3}+\frac{1}{24}]=20\sqrt{46}$.

(17pts.) **4.** Let the velocity field of a fluid be described by $F=xi+yj+zk$ (measured in meters per second). Compute how many cubic meters of fluid per second are crossing the surface of the sphere $x^2+y^2+z^2=1, z\ge 0$, where the normals are pointing up.

This is a surface integral of a vector valued function. The parameterization is $\Phi(\theta,\phi)=(\cos(\theta)\sin(\phi),\sin(\theta)\sin(\phi),\cos(\phi)), 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{2}$. The normal is $T_\theta\times T_\phi=(\cos(\theta)\sin^2(\phi),\sin(\theta)\sin^2(\phi),\cos(\phi)\sin(\phi))$, so the surface integral is $\int_0^{\pi/2}\int_0^{2\pi}(\cos(\theta)\sin(\phi),\sin$ $(\cos(\theta)\sin^2(\phi),\sin(\theta)\sin^2(\phi),\cos(\phi)\sin(\phi))d\theta d\phi=\int_0^{\pi/2}\int_0^{2\pi}\sin(\phi)d\theta d\phi=2\pi$.

(16pts.) **5.** Verify Green's Theorem for the disk D with center $(0,0)$ and radius R and the functions $P(x,y)=xy^2, Q(x,y)=-yx^2$.

The parameterization for the boundary is $x(t)=R\cos(t), y(t)=R\sin(t)$, so the line integral is $\int_C Pdx+Qdy=\int_0^{2\pi}(-R^4\cos(t)\sin^3(t)-R^4\cos^3(t)\sin(t))dt=-R^4\int_0^{2\pi}\cos(t)\sin(t)d$ $-R^4\sin^2(t)|_0^{2\pi}=0$. The double integral is $\int\int_D(Q_x-P_y)dA=\int\int_D(-2xy-2xy)dxdy=\int_0^1\int_0^{2\pi}-4r^2\cos(\theta)\sin(\theta)rdrd\theta=-\int_0^{2\pi}\cos(\theta)\sin(\theta)d\theta=0$. This shows that Green's Theorem is true for these functions.

(17pts.) **6.** Verify Stokes' Theorem for the function $F(x,y,z) = (2x\cos(y)e^z, -x^2\sin(y)e^z, x^2\cos(y)e^z)$ across the unit disk in the xy-plane centered at the origin. (Hint: Your answer to this problem should contain essentially no computations but a reasonable amount of explanation!)

The only computation needed in this case is $F(x,y,z) = \nabla f(x,y,z)$ for $f(x,y,z) = x^2\cos(y)e^z$. Once this is established, one part of Stokes' Theorem asks for a surface integral of the curl of the vector valued function F, but the curl of a gradient is 0, so the surface integral evaluates to 0. The other part of Stokes' Theorem asks for a line integral of the vector valued function around the boundary of the surface, but this is a simple closed curve line integral of a gradient. Line integrals of gradients are $f(c(b)) - f(c(a))$, and $c(b) = c(a)$, so $f(c(b)) - f(c(a)) = 0$, so the line integral is also 0. The two agree as expected.

Have a GREAT Thanksgiving!!