Math 350
Spring, 2000

HOMEWORK #2

Do 50 points of the following problems (due 1/25/00).

15 pts. **1** Construct the following binary codes if possible, or explain why it is not possible. (see problem 1, p.27)

a. (6,32,2)

Take all distinct binary 5-tuples and add a parity check bit on the end.

b. $(k, 3, k-1)$

From work in class, if such a code existed, it is equivalent to a code with the all 0 codeword. If the all 0 codeword is in the code, then any other codeword would need to have weight at least $k-1$. If we had two such codewords of weight $k-1$, then the distance between those two codewords is at most 2. Thus, unless $k \leq 3$, this code is not possible. We can construct the code for $k = 1, 2$, and 3.

c. (6,15,3)

The sphere-packing bound will not allow this since $15(7) > 2^6 = 64$.

15 pts. **2** How many inequivalent binary (3,3,1) codes are there?

There are 3 inequivalent binary codes with these parameters: the equivalence classes are $\{\{000, 001, 010\}, \{000, 001, 100\}, \{000, 100, 010\}, \{000, 001, 011\}, \{000, 011, 010\}, \{000, 110, 010\}, \{000, 110, 100\}, \{000, 101, 100\}, \{000, 101, 001\}\}$; $\{\{000, 001, 110\}, \{000, 001, 111\}, \{000, 101, 010\}, \{000, 111, 010\}, \{000, 100, 011\}, \{000, 100, 111\}, \{000, 111, 011\}, \{000, 110, 111\}, \{000, 101, 111\}\}$; $\{\{000, 011, 110\}, \{000, 011, 101\}, \{000, 101, 110\}\}$. This takes care of all possibilities because we only needed to check $\binom{7}{2} = 21$ possible codes once we required the all 0 codeword to be in the code.

20 pts. **3** Write out the spheres of radius 1 are for the code listed at the bottom of page 23 (there are 128 elements of $(F_2)^7$, and each of these elements should be in a sphere around a codeword). Explain how this list illustrates the sphere packing bound.

$\{0000000; 0000001, 0000010, 0000100, 0001000, 0010000, 0100000, 1000000\}$;
$\{1111111; 1111110, 1111101, 1111011, 1110111, 1101111, 1011111, 0111111\}$;
$\{1011000; 1011001, 1011010, 1011100, 1010000, 1001000, 1111000, 0011000\}$;
$\{0101100; 0101101, 0101110, 0101000, 0100100, 0111100, 0001100, 1101100\}$;
$\{0010110; 0010111, 0010100, 0010010, 0011110, 0000110, 0110110, 1010110\}$;
$\{0001011; 0001010, 0001001, 0001111, 0000011, 0011011, 0101011, 1001011\}$;
$\{1000101; 1000100, 1000111, 1000001, 1001101, 1010101, 1100101, 0000101\}$;
$\{1100010; 1100011, 1100000, 1100110, 1101010, 1110010, 1000010, 0100010\}$;
$\{0110001; 0110000, 0110011, 0110101, 0111001, 0100001, 0010001, 1110001\}$;
$\{0100111; 0100110, 0100101, 0100011, 0101111, 0110111, 0000111, 1100111\}$;
$\{1010011; 1010010, 1010001, 1010111, 1011011, 1000011, 1110011, 0010011\}$;
$\{1101001; 1101000, 1101011, 1101101, 1100001, 1111001, 1001001, 0101001\}$;
$\{1110100; 1110101, 1110110, 1110000, 1111100, 1100100, 1010100, 0110100\}$;
$\{0111010; 0111011, 0111000, 0111110, 0110010, 0101010, 0011010, 1111010\}$;
$\{0011101; 0011100, 0011111, 0011001, 0010101, 0001101, 0111101, 1011101\}$;
$\{1001110; 1001111, 1001100, 1001010, 1000110, 1011110, 1101110, 0001110\}$.

This illustrates the sphere packing bound because we have 16 spheres, each containing 8 strings of length 7. None of the spheres overlap, and the total number of strings does not exceed the number of possible strings, namely $2^7 = 128$. In this case, every possible string occurs in exactly one sphere.

35 pts., ⋆ **4** Let D be the design described as follows: let the points be the squares of the 4 x 4 square shown below, and let the blocks be made up of the 6 points on the same row or column as some square. Thus, there are 16 points and 16 blocks, and I claim that these points and blocks form a (16,16,6,6,2)-design: show this. Form the incidence matrix of this structure, and let the code be all of the rows of this matrix as well as all possible sums of the rows. Find M and d for this code.

	x		

x = point

	y		
	y		
y		y	y
	y		

y = points on block

This incidence structure clearly has 16 points (the squares in the 4×4 grid). The 16 blocks are "centered" at these points. Each block clearly has 6 points on it, and each point is on 6 blocks whose centers are the other squares in the same row and column. If you choose blocks from the same row (or column), then their intersection will be the other two points in that row (or column). If the blocks are from different rows and columns, their intersection will be the other two points in the rectangle determined by the centers of the blocks. Similarly, any pair of points in the same row (or column) will be on the blocks centered at the other two points in the row (or column). Any pair of points in different rows and columns will be on the blocks whose centers are the other corners of the rectangle determined by the points. This establishes the fact that this is a design.

To calculate the number of codewords, we start with the 16 blocks of the design (all of weight 6). If we sum up the 4 blocks whose centers are in the same row (or column), we get the all 1 codeword. The all 0 codeword is achieved by adding any block to itself. When we add the all 1 codeword to the blocks, we get 16 codewords of weight 10 (the complements of the blocks). If we add two distinct blocks together, we get codewords of weight 8 (there are 2 points in common, leaving 4 points on each block without a match in the other). There are $\binom{16}{2}$ possible ways to add blocks, but in each case 4 of those will yield the same word of weight 8, so there are $\frac{\binom{16}{2}}{4} = 30$ words of weight 8. Adding these up, that makes 64 codewords. Finally, we argue that the sum of any 3 blocks is either a block or the complement of a block. This requires a tedious case-by-case analysis (which I leave to the reader!) unless someone had a more clever approach.