Test # 1

Take home test (closed book, 2:30 hours)

Davis
M245

Name:
Pledge:

10pts **1.** Is it possible for a system of 3 equations with 4 unknowns to have a unique solution? No solution? Infinite number of solutions? Explain your answers.

It is not possible for a system of 3 equations with 4 unknowns to have a unique solution: there must be at least one free variable, so if the system is consistent it has an infinite number of solutions. An example of a system with no solutions is $x_1 = 0; x_1 = 1; x_2 + x_3 + x_4 = 0$ (as long as there are two equations in the system, it is always possible to get an inconsistent system). An example of a system with an infinite number of solutions is $x_1 = 0; x_2 = 0; x_3 = 0$ (in this case, x_4 is free and solutions have the form $x_4(0, 0, 0, 1)$).

10pts **2.** Consider the matrix $\begin{pmatrix} 0 & 4 & -3 & 2 \\ 3 & 0 & 1 & 3 \\ 2 & -5 & 1 & 4 \end{pmatrix}$. Row reduce the matrix to reduced echelon form. Is the vector $(4, 2, 1)$ in the span of the columns of the matrix? Explain your answer.

The reduced echelon form is $\begin{pmatrix} 1 & 0 & 0 & \frac{59}{41} \\ 0 & 1 & 0 & \frac{-20}{41} \\ 0 & 0 & 1 & \frac{-54}{41} \end{pmatrix}$.

Since there is a pivot in every row, every vector in R^3 is in the span of the columns of the matrix, so $(4, 2, 1)$ is in the span (note that you do NOT have to find the linear combination of the columns that make this true, although that would be an acceptable way to handle this).

10pts **3.** Describe the solution set for the homogeneous system of equations

$$\begin{aligned} x_1 - x_2 + 3x_3 - 3x_4 &= 0 \\ 2x_1 - 2x_2 + 7x_3 - 7x_4 &= 0 \\ 3x_1 - 3x_2 + 10x_3 - 10x_4 &= 0 \end{aligned}$$

The reduced echelon form of the matrix is $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so the solution set of the system is $x_2(1,1,0,0) + x_4(0,0,1,1)$ (x_2 and x_4 are free variables).

Compare it with the solution set for the system

$$\begin{aligned} x_1 - x_2 + 3x_3 - 3x_4 &= 3 \\ 2x_1 - 2x_2 + 7x_3 - 7x_4 &= 7 \\ 3x_1 - 3x_2 + 10x_3 - 10x_4 &= 10 \end{aligned}$$

In order to find the solution set for this system, we simply need to find a particular solution and add that to the homogeneous solution from the first part of the problem. One solution to this problem is $(0,0,1,0)$, so the general solution is $(0,0,1,0) + x_2(1,1,0,0) + x_4(0,0,1,1)$.

10pts **4.** Show that the solutions to the nonhomogeneous system of equations $A\mathbf{x} = b$ all have the form $\mathbf{x} = p + v_h$ where $Ap = b$ and $Av_h = 0$.

Suppose $\mathbf{x}$ is a solution to the system $A\mathbf{x} = b$. Then $A\mathbf{x} = Ap$, which implies that $A(\mathbf{x} - p) = 0$. If we set $v_h = \mathbf{x} - p$, we see that $\mathbf{x} = p + v_h$ as required (every solution is the particular solution plus a homogeneous solution.).

10pts **5.** Use Mathematica to decide if the linear transformation T defined by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 15 & -2 & 0 & 13 & 7 \\ -5 & 12 & 19 & -25 & 8 \\ 22 & 33 & 44 & 55 & 9 \end{pmatrix}$, is an onto transformation (make sure you include the definition of onto in your answer to this question).

A transformation is called *onto* if for every element b of R^m, there exists an element $\mathbf{x}$ of R^n so that $T(\mathbf{x}) = b$. This is equivalent to saying that every row of the matrix has a pivot. There are many ways to use Mathematica to determine the answer to this question: perhaps the easiest way to do that is to use the command RowReduce[M].

10pts **6.** Sketch the image of the house after it is transformed by the following linear transformations (start with the picture of the house in the first quadrant as we did in class):

A. $T(\mathbf{x}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}$

This matrix flips the house around the y-axis. It should look as though the door of the house is on the right hand side.

B. $T(\mathbf{x}) = \begin{pmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

This transformation turns the house on its side, stretches the floor, shrinks the walls, and shifts the house up 1 and over 1.

C. $T(\mathbf{x}) = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \mathbf{x}$

The floor is pointing down, and the walls are pointing across.

10pts **7.** If u and v are not scalar multiples of eachother and if $\{u, v, w\}$ forms a linearly dependent set of vectors, show that $w \in \text{Span}\{u, v\}$.

Let $au + bv + cw = 0$ be a linear combination of the vectors $\{u, v, w\}$ where not all of a, b, and c are 0. It is not possible for $c = 0$ since if that was the case, $au = -bv$ and u and v would be scalar multiples of eachother. Thus, $w = \frac{1}{c}(-au + bv)$ implying that $w \in \text{Span}\{u, v\}$.

10pts **8.** Given $A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}$, compute AA^T.

$$AA^T = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix}$$

Is AA^T invertible? If so, find an inverse; if not, explain why not.

AA^T is invertible since $ad - bc = 200 \neq 0$. The inverse is $(AA^T)^{-1} = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{20} \end{pmatrix}$

BONUS PROBLEM: Compute $A^T A$, and determine whether it is invertible or not (you must explain your answer to get the points!).

$$AA^T = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 4 & 0 & 8 \\ 3 & 0 & 9 & 0 \\ 0 & 8 & 0 & 16 \end{pmatrix}$$

This matrix is not invertible because it only has 2 pivots (just do a little row reduction). This means that there are an infinite number of solutions to the homogeneous system of equations, but that is impossible for an invertible matrix.

20pts **9.** Mark each of the following statements either True or False. Justify each answer.

A. An $m \times n$ coefficient matrix with a pivot in every row is equivalent to a matrix whose columns span R^n.

False. The columns are in R^m, not R^n, so they can't span.

B. If a system of equations does not have any free variables, then there is a unique solution.

False. $x_1 = 0; x_2 = 0; x_2 = 1$ is a system of 3 equations in 2 unknowns with NO solutions (any inconsistent system with no free variable would work here).

C. A linear transformation T must satisfy $T(0) = 0$.

True. By properties of linear transformations, $T(0) = T(0+0) = T(0) + T(0)$. Subtracting $T(0)$ from both sides yields $T(0) = 0$.

D. If the first column of the matrix AB is all 0, then the first column of B must be all 0.

False. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$. Then $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ but the first column of B is not 0.

E. If the columns of an $n \times n$ matrix span R^n, then the columns must be linearly independent.

True. In order to span R^n, the matrix must have a pivot in every row. This implies that there is a pivot in every column, so there are no free variables and hence the columns must be linearly independent.