Test # 2

Take home test (closed book, 2:30 hours)

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M245

Name:
Pledge:

10 pts. **I.** Find the 3×3 matrix that rotates points in the plane by 120 degrees about the point (2,-5) using homogeneous coordinates. Briefly explain why we use homogeneous coordinates to do computer graphics.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

Homogeneous coordinates are used so translations will be linear transformations.

10pts. **II.** Find a basis for $NulA, ColA$, and $RowA$ of the following matrix (you may use Mathematica if you wish). State the rank theorem and verify it for this matrix.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 & 4 \\ 2 & 4 & 6 & 8 & 0 & 0 \\ 3 & 6 & 3 & 4 & 5 & 6 \\ 4 & 8 & 12 & 16 & 22 & 30 \end{pmatrix}$$

The reduced echelon form of this matrix is

$$B = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

A basis for the row space is either the 4 rows A or B; A basis for the column space is any 4 linearly independent vectors in R^4: it is most natural to take the pivot columns from A (columns 1,3,5,6); A basis for the Null Space is [-2 1 0 0 0 0] and [0 0 -4/3 1 0 0]. The Rank Theorem states that the Rank of the matrix plus the dimension of the null space is the number of columns. In this case, the rank is 4, the dimension of the null space is 2, and the number of columns is 6.

10pts. **III.** Let V be a vector space and let $\{v_1, v_2, \ldots, v_p\}$ be a basis for V. Show that every element of V can be written uniquely as a linear combination of the vectors $\{v_1, v_2, \ldots, v_p\}$.

Since $\{v_1, v_2, \ldots, v_p\}$ is a basis for V, every vector in V is contained in the span of $\{v_1, v_2, \ldots, v_p\}$. Thus, we only need to show uniqueness. Suppose $v = a_1 v_1 + \ldots + a_p v_p$ and $v = b_1 v_1 + \ldots + b_p v_p$. Setting the linear combinations equal and moving them to one side, we get $(a_1 - b_1)v_1 + \ldots + (a_p - b_p)v_p = 0$. Since $\{v_1, v_2, \ldots, v_p\}$ is a basis, the vectors are linearly independent, and that implies that $a_1 - b_1 = 0, \ldots, a_p - b_p = 0$. This demonstrates that there was really only one way to write the vector v, proving the result.

10pts. **IV.** Let V be the collection of polynomials of degree at most 6 that are divisible by the polynomial x^2+1. Either verify that V is a subspace, or give a counterexample why it is not a subspace.

We first need a clear understanding of what it means for a polynomial to be divisible by x^2+1: $f(x)$ is divisible by x^2+1 if there is a $g(x)$ so that $f(x) = (x^2+1)g(x)$. Suppose $f_1(x)$ and $f_2(x)$ are both divisible by x^2+1 with associated polynomials $g_1(x)$ and $g_2(x)$ respectively. Then $f_1(x)+f_2(x) = (x^2+1)(g_1(x)+g_2(x))$, so $f_1(x)+f_2(x)$ are both in the collection of polynomials we are considering. A similar argument shows that this set is closed under scalar multiplication ($cf(x) = (x^2+1)(cg(x))$), so this set is a subspace.

10pts. **V.** Compute the determinant of the matrices $E(n)$, where $E(n)$ is the matrix with 1's down the main diagonal and 1's on the diagonals to either side of the main diagonal, for $n = 2,3,4,5$ and 6. For example, $E(5) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. Do not use Mathematica for this problem, and explain what properties of determinants you used to answer this question.

$|E_2| = 0; |E_3| = -1; |E_4| = -1; |E_5| = 0; |E_6| = 1$. If you do cofactor expansion about the first column, you will see that in general $|E_n| = |E_{n-1}| - |E_{n-2}|$. This leads to the pattern 0,-1,-1,0,1,1, 0,-1,-1, …, and from this we get the Bonus question of $|E_{100}| = -1$.

Bonus problem: Find the determinant of $E(100)$, and justify your answer.

10pts. **VI.** Use the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to verify that

A. Row replacement does not change the determinant of a matrix.

$$\begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix} = a(d+kb) - b(c+ka) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

B. Interchange multiplies the determinant by -1.

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad-bc) = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

C. Scaling a row by k multiplies the determinant by k.

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad - kbc = k(ad-bc) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

10pts. **VII.** Let $\mathcal{B} = \{\begin{bmatrix} 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\}$ and $\mathcal{C} = \{\begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix}\}$ be bases for R^2. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$. If $\mathbf{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{C}}$, and show how the change of coordinates matrix takes you from one to the other.

To find the change of coordinates matrix, the easiest way is to form the matrix $\begin{pmatrix} -1 & 0 & 5 & 4 \\ 3 & -2 & 2 & 1 \end{pmatrix}$ and row reduce it to $\begin{pmatrix} 1 & 0 & -5 & -4 \\ 0 & 1 & -17/2 & -13/2 \end{pmatrix}$. The second part of this matrix is the change of coordinates matrix. By setting up the augmented matrix, you can get $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 19/3 \\ -20/3 \end{bmatrix}$ and $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -5 \\ -21/2 \end{bmatrix}$

10pts. **VIII.** Use Mathematica to find a basis for the space spanned by the following vectors: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 8 \\ 2 \end{bmatrix},$

Is this space equal to R^4? Explain your answer.

The first 4 vectors are linearly independent, and they form a basis for the space spanned by all of the vectors. Since this is a 4-dimensional subspace of R^4, it must be all of R^4.

20pts. **IX.** Answer the following true/false questions. In each case, either give a brief (two complete sentences) justification for your answer or provide a counterexample.

A. If B is an echelon form of a matrix A, then the pivot columns of B form a basis for $ColA$.

False. The matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ serves as a counterexample.

B. If the columns of an $n \times n$ matrix are linearly independent, then the determinant of that matrix is 0.

False. The $n \times n$ identity matrix has linearly independent columns, and its determinant is 1.

C. A subset H of a vector space V is a subspace of V if the zero vector is in H.

False. Take the first quadrant of the plane together with the axes as a counterexample.

D. Let $\mathcal{B}$ be a basis for R^n. If $P_{\mathcal{B}}$ is the change-of-coordinates matrix, then $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$ for any $\mathbf{x} \in R^n$.

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True. The vector $[\mathbf{x}]_{\mathcal{B}}$ contains the weights to use for the basis, and the matrix multiplication simply gives the linear combination of the basis vectors which yields $\mathbf{x}$.

E. If $\{v_1, \ldots, v_p\}$ is a spanning set for a vector space V, then $dimV \geq p$.

False

Take any 3 vectors in R^2.